

# BRST Cohomology and Renormalizability of Quantum Gravity near Two Dimensions

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## Abstract

We discuss the renormalizability of quantum gravity near two dimensions based on the results obtained by a computation of the BRST-antibracket cohomology in the space of local functionals of the fields and antifields. We justify the assumption on the general structure of the counterterms which have been used in the original proof of renormalizability of the quantum gravity near two dimensions.

# 1 Introduction

Recently, the renormalizability of the quantum gravity near two dimensions has been actively studied [1][2][3]. The authors in ref.[2] have proven that all necessary counterterms which may be required in the perturbative calculation can be supplied by the bare action which is invariant under the full diffeomorphisms. The gauge invariance imposes very strong constraints on the possible counterterms which are required to cancel the short distance divergences. However the conformal invariance is well known to be anomalous in two dimensional quantum gravity. Nevertheless the structure of the conformal anomaly also imposes strong constraints on the possible divergences. In fact it is assumed that the possible counterterm which is compatible with the anomaly is unique in [2] . proof of To provide the justification for this crucial assumption on the uniqueness of the counterterms is a main motivation of this paper.

The authors in the ref.[2] start with the tree level action which possesses the volume preserving diffeomorphism invariance. In order to recover the full diffeomorphisms invariance, they further require that the theory is independent of the background metric. As they argue, this requirement leads us to search a theory which is conformally invariant with respect to the background metric. Obviously the Einstein action is such a theory. They finally conjecture that the requirement of the background independence leads us uniquely to the Einstein action which is invariant under the diffeomorphisms, as the bare action. We will try to justify the conjecture on the ground of mathematical and model independent analysis in this paper.

It has been recognized that the BRST method provides us one of the most powerful tools for quantizing theories with local gauge symmetries [4]. The original BRST method has been extended further. One of the extended BRST quantization methods is the field-antifield formalism developed by Batalin-Vilkovisky [5] [6]. In this formulation, an odd symplectic structure which is called antibracket plays an important role. The BRST-antibracket cohomology of the two-dimensional Weyl invariant gravity theory has been completely computed in the space

of local functionals of the fields and antifields in ref.[10]. The starting point of the analysis used in that work is the field content, and the symmetry transformations. These include the diffeomorphisms and the Weyl transformations.

These symmetries are realized on the scalar matter fields and on the two-dimensional metric:

$$\begin{aligned}
\delta X^\mu &= \xi^\rho \partial_\rho X^\mu, \\
\delta g_{\alpha\beta} &= \xi^\rho \partial_\rho g_{\alpha\beta} + g_{\alpha\rho} \partial_\beta \xi^\rho + g_{\beta\rho} \partial_\alpha \xi^\rho + C g_{\alpha\beta}, \\
\delta \xi^\alpha &= \xi^\rho \partial_\rho \xi^\alpha, \\
\delta C &= \xi^\rho \partial_\rho C,
\end{aligned} \tag{1}$$

where  $C$  and  $\xi^\alpha$  are respectively the Weyl and diffeomorphism ghosts and  $\delta$  is the BRST transformation.

provided by diffeomorphism ghosts, In the language of the BRST cohomology, generally, the anomalies (Weyl anomaly, gauge anomaly, etc...) are represented by cohomology classes with ghost number one. It can be used to compute other classes with the different values of the ghost number. In particular the class with ghost number zero is interesting since it contains generic classical actions. This gives the possibility to construct the most general classical action by computing the class with ghost number zero for the desired field content and gauge invariances. polynomial in derivatives of all

Here we briefly summarize the results [10] which will be mainly used in this paper. In the ghost number zero sector, the BRST cohomology without antifields determines the most general classical action:

$$\frac{1}{2} \int d^2x (\sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)), \tag{2}$$

where  $G_{\mu\nu}(X)$ ,  $B_{\mu\nu}(X)$  are arbitrary functions of  $X^\mu$  that are respectively symmetric and anti-symmetric under  $\mu \leftrightarrow \nu$ , and  $\epsilon^{\alpha\beta}$  is the constant antisymmetric tensor. The BRST cohomology in the ghost number one sector gives the candidates of anomalies. It seems reasonable to as-

sume that the anomalies are left-right symmetric since the action (2) is left-right symmetric. Accordingly the candidates of anomalies in two dimensional conformal gravity are

$$\int d^2x C \sqrt{g} R, \quad (3)$$

$$\frac{1}{2} \int d^2x C (\sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu f_{(\mu\nu)}(X) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu f_{[\mu\nu]}(X)), \quad (4)$$

where  $f_{(\mu\nu)}(X)$  and  $f_{[\mu\nu]}(X)$  are arbitrary functions of the matter fields  $X^\mu$ . symmetrization of  $\mu, \nu$ . We recall here that the diffeomorphism anomaly and the Weyl anomaly are cohomologically equivalent. Therefore we have a choice to respect either the diffeomorphism invariance or the conformal invariance but not the both. We choose to respect the diffeomorphism invariance when we quantize the theory.

The organization of this paper is as follows. In section two, we set up a model which can preserve the diffeomorphism and the Weyl invariance in the  $2 + \varepsilon$  dimensions. Then we briefly review the proof of the renormalizability of  $2 + \varepsilon$  dimensional quantum gravity. In section three, we prove the uniqueness of the counterterms based on the BRST cohomology which justifies the assumption used in [2]. This proof of the uniqueness of the counterterms is the essential part of this paper. We conclude in section four with discussions.

## 2 Quantum gravity in $2 + \varepsilon$ dimensions

The classical action which possesses the diffeomorphism and the Weyl invariance in the  $2 + \varepsilon$  dimensions is described by

$$\begin{aligned} S = & \frac{1}{2} \int d^{2+\varepsilon}x \sqrt{g} (\partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} G_{\mu\nu}(X) \\ & + \epsilon^{ab} e_a^\alpha e_b^\beta \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)) (1 + \frac{1}{2} \sqrt{\frac{\varepsilon}{2(D-1)}} \psi)^2 \\ & + \int d^{2+\varepsilon}x \sqrt{g} \{ R(1 + \sqrt{\frac{\varepsilon}{2(D-1)}} \psi + \frac{\varepsilon}{8(D-1)} \psi^2) - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi g^{\mu\nu} \}, \end{aligned} \quad (5)$$

where  $X^\mu$  ( $\mu = 0, \dots, D-1$ ) is a set of scalar matter fields, and  $\psi$  is another scalar field.  $G_{\mu\nu}$  and  $B_{\mu\nu}$  ( $\mu = 0, \dots, D-1$ ) are, respectively, arbitrary functions of the  $X^\nu$  satisfying

$$G_{\mu\nu} = G_{\nu\mu}, \quad B_{\mu\nu} = -B_{\nu\mu}, \quad (6)$$

and  $g^{\alpha\beta}$  are the metric on the  $2 + \varepsilon$  dimensional world sheet. The "twei"-bein field which is a connection between the world sheet and the tangent space is denoted by  $e_a^\alpha$ , the index  $a$  refers to the local Lorentz transformation on the tangent space.  $\epsilon^{ab}$  is a constant antisymmetric tensor on the  $2 + \varepsilon$  dimensional tangent space of the world sheet [12].

At the limit of  $\varepsilon \rightarrow 0$ , the form of (5) agrees with the form of (2) which is obtained by considering the BRST cohomology with ghost number zero. In this identification, the following Einstein term

$$\int d^2x \sqrt{g} R \quad (7)$$

can be ignored as the surface term.

Next we recall the standard perturbative evaluation of the effective action by  $\hbar$  expansion. Due to the BRST invariance, the effective action  $\Gamma$  satisfies the following Ward-Takahashi identity:

$$\Gamma * \Gamma = 0, \quad (8)$$

It is well-known that the Eq.(8) imposes strong restriction on the possible divergences which may appear in the perturbative evaluation of the effective action.

The effective action may be expanded in  $\hbar$  as

$$\Gamma = S + \hbar \Gamma^{(1)} + \hbar^2 \Gamma^{(2)} + \dots, \quad (9)$$

where  $S$  is the classical action and  $\Gamma^{(n)}$  ( $n = 1, 2, \dots$ ) are the quantum corrections of the theory. Substituting Eq.(9) into Eq.(8), one obtains

$$S * S = 0, \quad (10)$$

$$S * \Gamma^{(1)} = 0, \quad (11)$$

$$\Gamma^{(1)} * \Gamma^{(1)} + 2S * \Gamma^{(2)} = 0, \quad (12)$$

.....

Here the equation (10),  $S * S = 0$ , is trivially satisfied if the classical action  $S$  preserve the gauge symmetries. The equation (11) constrains the possible structure of the counter terms at the one loop level. In fact we prove that the structure of the counter terms are unique in the next section. As it will be shown there, we can iterate this procedure order by order in the  $\hbar$  expansion to all orders and the proof of the renormalizability follows inductively.

### 3 Uniqueness of the counterterms

Our task is to determine the possible counterterms of the theory which is consistent with the equation (11). The effective action  $\Gamma^{(1)}$  at the one-loop may be divided into the finite and infinite parts, which are denoted by  $\Gamma_{fin.}$  and  $\Gamma_{div.}$  respectively:

$$\Gamma^{(1)} = \Gamma_{fin.} + \Gamma_{div.} \quad (13)$$

The infinite part should be understood as the term which possesses the  $1/\varepsilon$  pole in this paper. The possible structure of  $\Gamma_{div.}$  is constrained by the WT identity due to the following considerations.

We find that  $S * \Gamma_{div.}$  is finite since  $S * \Gamma_{fin.}$  is finite. By using Eq.(10) and the Jacobi identity on the operation  $*$ , we also find that  $S * (S * \Gamma_{div.}) = 0$ . Hence we conclude that  $S * \Gamma_{div.}$  is nothing else but an anomaly if it is nonvanishing. If the gauge symmetry is independent of the regularization parameter, we can conclude that  $S * \Gamma_{div.} = 0$ . It is indeed the case for QED or QCD. These theories certainly do not possess anomalies. However our gauge transformation (conformal transformation) depends on the  $\varepsilon$  explicitly. Therefore we cannot conclude that  $S * \Gamma_{div.} = 0$ . However we can still conclude that  $S * \Gamma_{fin.}$  is finite and must be consistent with the anomaly.

As we have already known all anomalies from the BRST cohomological analysis, we can uniquely determine the possible divergent part  $\Gamma_{div.}$  as follows:

$$\frac{2}{\varepsilon} \int d^{2+\varepsilon} x \sqrt{g} R, \quad (14)$$

$$\frac{1}{\varepsilon} \int d^{2+\varepsilon} x \sqrt{g} (g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu f_{(\mu\nu)}(X) + \epsilon^{ab} e_a^\alpha e_b^\beta \partial_\alpha X^\mu \partial_\beta X^\nu f_{[\mu\nu]}(X)). \quad (15)$$

It is because

$$\delta \left( \frac{2}{\varepsilon} \int d^{2+\varepsilon} x \sqrt{g} R \right) = \int d^2 x C \sqrt{g} R, \quad (16)$$

$$\begin{aligned} & \delta \left( \frac{1}{\varepsilon} \int d^{2+\varepsilon} x \sqrt{g} (g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu f_{(\mu\nu)}(X) + \epsilon^{ab} e_a^\alpha e_b^\beta \partial_\alpha X^\mu \partial_\beta X^\nu f_{[\mu\nu]}(X)) \right) \\ &= \frac{1}{2} \int d^2 x C (\sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu f_{(\mu\nu)}(X) + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu f_{[\mu\nu]}(X)), \end{aligned} \quad (17)$$

where  $\delta$  means the BRST transformation. On the right-hand side of these equations, we have taken the  $\varepsilon \rightarrow 0$  limit since such a limit is well defined. Indeed they become the anomalies which are shown in the equations (3) and (4) in such a limit.

What we have shown here is that these divergences are consistent with the anomalies. In order to prove the uniqueness of the counter terms, let us suppose that there are two different solutions for  $\Gamma_{div.}$ . Then the difference of the two solutions  $\Delta(\Gamma_{div.})$  satisfies  $S * \Delta(\Gamma_{div.}) = 0$ . Therefore the possible freedom of the solution is the addition of such divergences which satisfy  $S * \Delta(\Gamma_{div.}) = 0$ . However this freedom is also constrained by the BRST cohomology analysis with the ghost number zero. The only such a solution is of the original action type as we have listed in the equation (2) in the introduction. These divergences can be canceled by the renormalization of the couplings and the fields. Of course we may also have BRST trivial divergences. However they can be renormalized by the wave function and the gauge fixing part. Due to the triviality of the renormalization of the divergences which satisfy  $S * \Gamma_{div.} = 0$ , it is sufficient to consider the anomaly type divergences to prove the renormalizability. This completes the proof of the uniqueness of the nontrivial counter terms of the quantum gravity in  $2 + \varepsilon$  dimensions.

In the remaining part of this paper, we impose the following global symmetry on the matter  $X^\mu$  field:

$$X^\mu \longrightarrow X^\mu + C^\mu, \quad (18)$$

where  $C^\mu$  is a constant vector. By imposing the above global symmetry, one can kill the existence of the counterterms which contain arbitrary functionals of  $X^\mu$ ,  $G_{\mu\nu}$  and  $B_{\mu\nu}$ . In this way, we restrict our considerations to the matter  $X^\mu$  independent Weyl anomaly.

Let us consider the classical action

$$S = \int d^{2+\varepsilon}x \sqrt{g} (R\psi^2 \frac{\varepsilon}{8(D-1)} - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi g^{\mu\nu}). \quad (19)$$

(19) is invariant under the following BRST transformations (the infinitesimal gauge transformations):

$$\begin{aligned} \delta g_{\alpha\beta} &= \xi^\rho \partial_\rho g_{\alpha\beta} + g_{\alpha\rho} \partial_\beta \xi^\rho + g_{\beta\rho} \partial_\alpha \xi^\rho + C g_{\alpha\beta}, \\ \delta \xi^\alpha &= \xi^\rho \partial_\rho \xi^\alpha, \\ \delta C &= \xi^\rho \partial_\rho C, \\ \delta \psi &= \xi^\rho \partial_\rho \psi - C \frac{\varepsilon}{4} \psi. \end{aligned} \quad (20)$$

(19) can be transformed into the matter  $X^\mu$  independent part in (5) by the following translation:

$$\psi \rightarrow \psi + 2\sqrt{\frac{2(D-1)}{\varepsilon}}. \quad (21)$$

There emerges the following BRST nontrivial divergence at the one loop level:

$$-\frac{2}{\varepsilon} \int d^{2+\varepsilon}x \lambda \sqrt{g} R, \quad (22)$$

where  $\lambda$  is a known constant. The reason why there is such a divergence at the one loop level is that it is the only possible nontrivial candidate as we have listed in the equation (14). On the other hand, let  $\psi \rightarrow \psi + (4/\varepsilon)\tau$  in (19), then

$$\int d^{2+\varepsilon}x \sqrt{g} (R\psi^2 \frac{\varepsilon}{8(D-1)} - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi g^{\mu\nu}) + \tau \int d^{2+\varepsilon}x \psi \sqrt{g} R + \frac{2}{\varepsilon} \tau^2 \int d^{2+\varepsilon}x \sqrt{g} R. \quad (23)$$



If  $\tau$  is put to  $\sqrt{\lambda}$ , the divergence (22) is canceled by the last term of (23). After the cancellation, we obtain the following action:

$$S + \tau S' = \int d^{2+\varepsilon}x \sqrt{g} (R\psi^2 \frac{\varepsilon}{8(D-1)} - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi g^{\mu\nu}) + \tau \int d^{2+\varepsilon}x \psi \sqrt{g} R, \quad (24)$$

This action is the familiar linear dilaton type. The BRST transformation with respect to  $\psi$  field becomes:

$$\delta\psi = \xi^\rho \partial_\rho \psi - C \frac{\varepsilon}{4} \psi - \tau C. \quad (25)$$

Since  $\lambda$  is  $O(\hbar)$ , we regard  $\tau$  to be  $O(\sqrt{\hbar})$ . Therefore one may expand the effective action as follows:

$$\Gamma = S + \tau S' + \hbar \Gamma^{(1)} + \hbar \tau \Gamma^{(1)'} + \hbar^2 \Gamma^{(2)} + \hbar^2 \tau \Gamma^{(2)'} + \dots \quad (26)$$

As the classical action  $S$  and the BRST transformation have the symmetry  $\psi \rightarrow -\psi$  and  $\tau \rightarrow -\tau$ , the effective action  $\Gamma$  is also invariant under them.  $\Gamma^{(n)'}$  do not contain the divergences of the form (22) since they have to contain odd numbers of  $\psi$  field due to the discrete symmetry.

Let us suppose that  $\Gamma$  is renormalizable up to order  $\hbar^{n-1}$ . At next order  $\hbar^n$ ,  $S * \Gamma^{(n)}$  must be a finite quantity since it is related to the finite quantity by the Ward-Takahashi identity. On the other hand,  $\Gamma^{(n)}$  may be generally split into a finite part  $\Gamma_{fin}^{(n)}$  and an infinite part  $\Gamma_{div}^{(n)}$ . Notice that  $S * \Gamma_{div}^{(n)}$  becomes a finite quantity since  $S * \Gamma^{(n)}$  is finite. Here one concludes again that  $S * \Gamma_{div}^{(n)}$  should be identified with an anomaly if it is nonvanishing since

$$S * (S * \Gamma_{div}^{(n)}) = 0. \quad (27)$$

$\Gamma_{div}^{(n)}$  should correspond to (??). Therefore we conclude that the nontrivial solution for  $\Gamma_{div}^{(n)}$  is also of (22) type.

Let us transform the action and the BRST transformations of the fields as follows,

$$\begin{aligned} & \int d^{2+\varepsilon}x \sqrt{g} (R\psi^2 \frac{\varepsilon}{8(D-1)} - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi g^{\mu\nu}) \\ & + (\tau + \tau_2 + \dots + \tau_n) \int d^{2+\varepsilon}x \psi \sqrt{g} R \\ & + \frac{2}{\varepsilon} (\tau + \tau_2 + \dots + \tau_n)^2 \int d^{2+\varepsilon}x \sqrt{g} R, \end{aligned} \quad (28)$$

$$\delta\psi = \xi^\rho \partial_\rho \psi - C \frac{\varepsilon}{4} \psi - (\tau + \tau_2 + \cdots + \tau_n) C \quad (29)$$

where  $\tau_n \sim \hbar^{n-1} \tau$ .

Supposing we find the BRST nontrivial divergence of the following form:

$$\hbar^n \Gamma_{div.}^{(n)} = -\frac{4}{\varepsilon} \lambda \int d^{2+\varepsilon} x \sqrt{g} R, \quad (30)$$

it can be renormalized by putting

$$\tau_n = \frac{\lambda - \tau_2 \tau_{n-1} \cdots}{\tau}, \quad (31)$$

at order  $\hbar^n$ . From these considerations, the renormalizability of the  $2 + \varepsilon$  gravity theory has been proved recursively.

## 4 Conclusions and discussions

We have given the justification of the assumption which played an important role in the original proof of the renormalizability of  $2 + \varepsilon$  dimensional quantum gravity [1] [2]. The analysis of the BRST antibracket cohomology which is independent of the model under given gauge symmetries and field content allows us to guarantee the uniqueness of the form of the counterterms.

In this paper, we have applied the candidates of anomalies in the two-dimensional Weyl invariant gravity theory [10] to the construction of the counterterms for the  $2 + \varepsilon$  dimensional gravity theory. By imposing the translation symmetry of  $X^\mu$  and considering the matter  $X^\mu$  independent part, we have proved the uniqueness of the counterterms which have been necessary to the proof of the renormalizability of the  $2 + \varepsilon$  dimensional gravity theory. Within this framework, we can show that the only possible BRST nontrivial divergence is (22) by using the analysis of the BRST antibracket cohomology.

Our investigation is still confined to the vanishing cosmological constant case. We may adopt the standard strategy to renormalize the cosmological constant operator for infinitesimally small cosmological constant. The cosmological constant operator explicitly breaks the conformal

invariance in two dimensions and hence is not usually considered in the BRST analysis of Weyl invariant gravity. Nevertheless we can construct such an operator which is invariant under the BRST transformation in  $2 + \varepsilon$  dimensions (20) and (25):

$$\int d^{2+\varepsilon}x \sqrt{g} (1 + \frac{\varepsilon}{4\tau} \psi)^{\frac{2D}{\varepsilon}} \sim \int d^{2+\varepsilon}x \sqrt{g} \exp(\psi/\tau) \quad (32)$$

Since the quantum correction  $\tau$  appears in the denominator in this expression, we need to sum the quantum corrections to all orders to obtain the scaling exponents of the cosmological constant operator and general gravitationally dressed operators. In fact it has been successful to reproduce the exact scaling exponents of two dimensional quantum gravity in this way. It has been argued that the bare cosmological constant operator can be renormalized in this form to all orders in [2]. Since this expression is unique to respect the BRST invariance of (20) and (25), we believe that such an argument can also be justified rigorously in the near future.

anomalies renormalizability.

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